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Mitteilung 2(1998)





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Abstract

In this paper a continuum based 3D–shell element for the nonlinear analysis of laminated shell structures is derived. The basis of the present finite element formulation is the standard 8–node brick element with tri–linear shape functions. Especially for thin structures under certain loading cases the displacement based element is too stiff and tends to locking. Therefore we use assumed natural strain and enhanced assumed strain methods to improve the relative poor element behaviour. The anisotropic material behaviour of layered shells is modeled using a linear elastic orthotropic material law in each layer. Linear and nonlinear examples show the applicability and effectivity of the element formulation.

Keywords: nonlinear 3D–shell formulation, composite material, enhanced assumed strains, assumed natural strains

1 Introduction

In structural mechanics the efficient computation of thin structures requires reliable and robust elements. In the past several shell elements have been developed, where the normal stresses in thickness direction have been included in the underlying variational formulation. For these types of elements a discretization of the reference surface is necessary. The nodal parameters are the displacement vector and the extensible director vector of the reference surface. The direct interpolation of the extensible director vector is proposed in several papers, see e.g. Büchter, Ramm, Roehl [1], Betsch, Stein [2], Eberlein, Wriggers [3], Bischoff, Ramm [4]. The multiplicative decomposition of the director field into a rotational part and a scalar stretching part has been investigated e.g. by Simo, Rifai, Fox [5], Betsch, Gruttmann, Stein [6] or Steinmann, Betsch, Stein [7]. The stresses are computed from a three-dimensional material law. This feature is especially useful for complicated nonlinear constitutive equations. The condition of vanishing thickness stresses for thin structures is approximately fulfilled within the weak formulation. Locking effects which occur when using a three-dimensional material law along with constant normal thickness strains can be avoided by application of the enhanced assumed strain method to the thickness strains, see [1].

The associated variational formulation of this method has been developed by Simo and Rifai [8]. For linear membrane elements the formulation is identical to the method of incompatible modes introduced by Wilson et al. [9]. Further aspects for linear applications are discussed in Andelfinger, Ramm [10], Korelc, Wriggers [11]. Geometrical nonlinearity has been included for two-dimensional problems by Simo and Armero [12] and for three-dimensional finite deformations by Simo, Armero, Taylor [13]. These elements are compared for finite plasticity with shell elements by Wriggers, Eberlein and Reese in [14]. Formulation in terms of the Green–Lagrangean strain tensor may be found in Li, Crook, Lyons [15], Betsch, Gruttmann, Stein [6] and Klinkel, Wagner [16].

However, for certain problems nodal degrees of freedom at the surface of the shell are more advantageous. Examples are deformation processes with contact and friction and the delamination problem of layered shells. For this purpose surface oriented shell elements have been developed e.g. by Schoop [17], Parisch [18], Miehe [19].

In this paper a continuum based 3D-shell element for laminated structures is derived. The essential features and novel aspects of the present formulation are summarized as follows.

- (i) The basis of the present element formulation is an eight-node brick element with tri-linear shape functions and displacement degrees of freedom. Thus, there are no rotational degrees of freedom and also no need for complicated update of a rotation tensor. In contrast to shell elements with rotational degrees of freedom no problems of soft or hard support will be encountered. Boundary conditions at the top or bottom surface of the brick-type shell element can be considered. Another advantage of the continuum based shell element is the compatibility of the data structure to three-dimensional CAD software which can be supported by standard mesh generators. Furthermore our element provides a correct description of the interlaminar shear and normal stresses with a sufficient mesh refinement in thickness direction. This is e.g. useful for a subsequent delamination analysis.
- (ii) The relative poor behaviour of a standard displacement element is improved using the assumed strain method and the enhanced strain method. Thus, the interpolation with special functions for the transverse shear strains and thickness strains requires a representation with convective coordinates. Due to this feature the element orientation has to be considered within the mesh generation. To avoid shear locking the assumed natural strain method introduced by Hughes, Tezduyar [20] and Bathe, Dvorkin [21] is applied to the transverse shear strains. Our numerical investigations show, that it is sufficient to define four collocation points in the middle plane of the element. Furthermore the thickness strains are approximated using special interpolation functions introduced in [2] for a shell formulation with an extensible director vector. This is necessary to avoid artificial thickness strains. Again for the present element, the interpolation is applied in the middle plane of the element.
- (iii) The membrane behaviour and the bending behaviour is essentially improved applying the enhanced assumed strain method to the membrane strains and to the thickness strains. The associated variational formulation is written in a Lagrangean setting using the Green–Lagrangean strain tensor. This yields a geometric stiffness matrix which is identical with the pure displacement formulation and therefore is simpler than the corresponding formulation in terms of the material deformation gradient, [19].
- (iv) With restriction to physical linear behaviour we implement a hyperelastic, orthotropic, three-dimensional constitutive equation of the St. Venant-Kirchhoff type. Hence the components of the constitutive tensor are given with respect to the convective coordinate system. In shell theory thickness integration of the stresses and linearized stresses yields the stress resultants and the shell stiffness. For layered structures one obtains the so-called laminate stiffness, see e.g. Wagner, Gruttmann, [22]. Unlike to [18] stress resultants are not introduced in this paper. Here, virtual work expressions and associated linearizations are integrated in thickness direction. This simplifies the finite element formulation. For the continuum based 3D-shell element a general integration algorithm for composite materials is applied.

The outline of the paper is as follows. In section 2–4 we present the kinematic, the material law, and the variational equations of a 3D–shell formulation with convective coordinates. The associated finite element is described in section 5. A special technique, which is necessary for integration through the layers in the 3D–case, is given in section 6. Finally we show in section 7 the applicability of the element formulation with four linear and nonlinear examples.

2 Kinematic of a convective 3D – Formulation

According to Fig. 1 convective coordinates are introduced to describe the stress and strain tensors. We consider thin shell structures with ξ^3 as thickness coordinate and ξ^1 and ξ^2 as inplane coordinates. Thus, the thickness strains, the transverse shear strains and the corresponding stress components are clearly defined. This is important for the subsequent finite element formulation.

Figure 1: Curvilinear coordinates and convective base vectors of the reference configuration and the current configuration

The position vectors of the reference configuration and the current configuration are denoted by \mathbf{X} and \mathbf{x} , respectively. The covariant base vectors are obtained by partial derivatives of the position vectors with respect to the convective coordinates ξ^i

$$\mathbf{G}_i = \frac{\partial \mathbf{X}}{\partial \xi^i}$$
, $\mathbf{g}_i = \frac{\partial \mathbf{x}}{\partial \xi^i}$, $i = 1, 2, 3$ (1)

whereas the contravariant base vectors are defined in a standard way by $\mathbf{G}_i \cdot \mathbf{G}^j = \delta_i^j$ and $\mathbf{g}_i \cdot \mathbf{g}^j = \delta_i^j$. Hence the deformation gradient is given by

$$\mathbf{F} = \mathbf{g}_i \otimes \mathbf{G}^i \quad . \tag{2}$$

This leads to the following representation of the Green–Lagrangean strain tensor

$$\mathbf{E} = E_{ij} \ \mathbf{G}^i \otimes \mathbf{G}^j \quad \text{with} \quad E_{ij} = \frac{1}{2} \left(g_{ij} - G_{ij} \right) \,. \tag{3}$$

Here $g_{ij} = \mathbf{g}_i \cdot \mathbf{g}_j$ and $G_{ij} = \mathbf{G}_i \cdot \mathbf{G}_j$ are the metric coefficients of the current configuration and of the reference configuration, respectively.

3 Material law

In this paper a linear relation between the second Piola–Kirchhoff stress tensor $\mathbf{S} = S^{ij} \mathbf{G}_i \otimes \mathbf{G}_j$ and the Green–Lagrangean strain tensor $\mathbf{E} = E_{ij} \mathbf{G}^i \otimes \mathbf{G}^j$ is postulated

$$\mathbf{S} = \mathbb{C} : \mathbf{E} . \tag{4}$$

The constitutive behaviour of laminated composites can be described using an orthotropic or transversal isotropic material law. For that purpose we define a local orthonormal basis system \mathbf{T}^i in the reference configuration, see Fig. 2. Here, \mathbf{T}^3 is the normal vector of the fibre plane and \mathbf{T}^1 describes the fibre direction.

Figure 2: Orthonormal base system \mathbf{T}^{i} in the reference configuration

The strain tensor **E** can be written with respect to the base systems \mathbf{G}^i and \mathbf{T}^i

$$\mathbf{E} = E_{ij} \mathbf{G}^i \otimes \mathbf{G}^j = \bar{E}_{ij} \mathbf{T}^i \otimes \mathbf{T}^j.$$
(5)

With $\mathbf{T}^i \cdot \mathbf{T}^k = \delta^{ik}$ we obtain the transformation

$$\bar{E}_{kl} = (\mathbf{T}^k \cdot \mathbf{G}^i) E_{ij} (\mathbf{G}^j \cdot \mathbf{T}^l) \,. \tag{6}$$

Following common usage in the finite element literature we order the components of ${\bf E}$ in a vector

$$\mathbf{E} = [E_{11}, E_{22}, E_{33}, 2E_{12}, 2E_{13}, 2E_{23}]^T,$$
(7)

thus equation (6) yields

$$\bar{\mathbf{E}} = \mathbf{T}_E \; \mathbf{E} \quad . \tag{8}$$

Introducing $t_{ki} = \mathbf{T}^k \cdot \mathbf{G}^i$ the transformation matrix \mathbf{T}_E is given by

$$\mathbf{T}_{E} = \begin{bmatrix} (t_{11})^{2} & (t_{12})^{2} & (t_{13})^{2} & t_{11}t_{12} & t_{11}t_{13} & t_{12}t_{13} \\ (t_{21})^{2} & (t_{22})^{2} & (t_{23})^{2} & t_{21}t_{22} & t_{21}t_{23} & t_{22}t_{23} \\ (t_{31})^{2} & (t_{32})^{2} & (t_{33})^{2} & t_{31}t_{32} & t_{31}t_{33} & t_{32}t_{33} \\ 2t_{11}t_{21} & 2t_{12}t_{22} & 2t_{13}t_{23} & t_{11}t_{22} + t_{12}t_{21} & t_{11}t_{23} + t_{13}t_{21} & t_{12}t_{23} + t_{13}t_{22} \\ 2t_{11}t_{31} & 2t_{12}t_{32} & 2t_{13}t_{33} & t_{11}t_{32} + t_{12}t_{31} & t_{11}t_{33} + t_{13}t_{31} & t_{12}t_{33} + t_{13}t_{32} \\ 2t_{21}t_{31} & 2t_{22}t_{32} & 2t_{23}t_{33} & t_{21}t_{32} + t_{22}t_{31} & t_{21}t_{33} + t_{23}t_{31} & t_{22}t_{33} + t_{23}t_{32} \end{bmatrix}$$

$$(9)$$

The constitutive matrix for orthotropic material behaviour is given here in the inverse form

$$\bar{\mathbb{C}}^{-1} = \begin{bmatrix} \frac{1}{E_1} & -\frac{\nu_{12}}{E_1} & -\frac{\nu_{13}}{E_1} & 0 & 0 & 0 \\ -\frac{\nu_{12}}{E_1} & \frac{1}{E_2} & -\frac{\nu_{23}}{E_2} & 0 & 0 & 0 \\ -\frac{\nu_{13}}{E_1} & -\frac{\nu_{23}}{E_2} & \frac{1}{E_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G_{12}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G_{13}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G_{23}} \end{bmatrix}$$
(10)

with the engineering constants E_i , G_{ij} and ν_{ij} . The components of $\bar{\mathbb{C}}^{-1}$ and $\bar{\mathbb{C}}$ refer to the orthonormal base system \mathbf{T}^i . For transversal isotropic material behaviour $E_2 = E_3$, $\nu_{12} = \nu_{13}$ and $G_{12} = G_{13}$ holds, if \mathbf{T}^1 is the normal vector of the plane of isotropy. Since the stored energy W_{0S} is an invariant quantity

$$W_{0S} = \frac{1}{2} \,\bar{\mathbf{E}}^T \bar{\mathbf{C}} \bar{\mathbf{E}} = \frac{1}{2} \,\mathbf{E}^T \mathbf{C} \mathbf{E} \tag{11}$$

holds. Here, \mathbb{C} is the constitutive matrix which refers to the convective coordinate system \mathbf{G}_i . Considering equation (8) one obtains

$$\mathbf{C} = \mathbf{T}_E^T \, \bar{\mathbf{C}} \, \mathbf{T}_E \,. \tag{12}$$

This completes the computation of the stress tensor with respect to the convective base system.

4 Variational formulation

The below presented enhanced assumed strain method is based on a three field variational formulation introduced by Simo, Rifai [8]. Within a geometrical nonlinear formulation Simo, Amero [12] and Simo, Amero, Taylor [13] applied this method using the enhanced displacement gradient. In contrast to that we follow the approach in [6] and [16]. Thus, the compatible Green–Lagrange strain tensor \mathbf{E} is enhanced as follows

$$\dot{\mathbf{E}} = \mathbf{E} + \dot{\mathbf{E}} \tag{13}$$

where $\tilde{\mathbf{E}}$ describes the enhanced part.

The body is loaded by surface loads $\hat{\mathbf{t}}$ and volume forces $\rho_0 \hat{\mathbf{b}}$. The variational framework for the enhanced assumed strain method is the following three field variational functional in a Lagrangean formulation

$$\Pi(\mathbf{u}, \tilde{\mathbf{E}}, \tilde{\mathbf{S}}) = \int_{\mathcal{B}_0} (W_{0S}(\mathbf{E} + \tilde{\mathbf{E}}) - \tilde{\mathbf{S}} : \tilde{\mathbf{E}}) \, \mathrm{d}V - \int_{\mathcal{B}_0} \rho_0 \hat{\mathbf{b}} \cdot \mathbf{u} \, \mathrm{d}V - \int_{\partial \mathcal{B}_0} \hat{\mathbf{t}} \cdot \mathbf{u} \, \mathrm{d}A \,.$$
(14)

The first term describes the internal potential, while the last two terms denote the potential of the external forces. Here \mathbf{u} , $\tilde{\mathbf{E}}$ and $\tilde{\mathbf{S}}$ are independent tensorial quantities. Thus, the first variation of the functional (14) is obtained via the directional derivative or the so-called Gâteaux derivative as

$$\delta \Pi = \int_{\mathcal{B}_0} \frac{\partial W_{0S}}{\partial \hat{\mathbf{E}}} : \delta \mathbf{E} \, \mathrm{d}V - \int_{\mathcal{B}_0} \rho_0 \hat{\mathbf{b}} \cdot \delta \mathbf{u} \, \mathrm{d}V - \int_{\partial \mathcal{B}_0} \hat{\mathbf{t}} \cdot \delta \mathbf{u} \, \mathrm{d}A$$
$$+ \int_{\mathcal{B}_0} (\frac{\partial W_{0S}}{\partial \hat{\mathbf{E}}} - \tilde{\mathbf{S}}) : \delta \tilde{\mathbf{E}} \, \mathrm{d}V$$
$$- \int_{\mathcal{B}_0} \delta \tilde{\mathbf{S}} : \tilde{\mathbf{E}} \, \mathrm{d}V \qquad = 0.$$
(15)

The first variation of the Green–Lagrangean strain tensor reads

$$\delta \mathbf{E} = \frac{1}{2} (\mathbf{g}_i \cdot \delta \mathbf{g}_j + \delta \mathbf{g}_i \cdot \mathbf{g}_j) \mathbf{G}^i \otimes \mathbf{G}^j .$$
⁽¹⁶⁾

Introducing the orthogonality condition for the stress field $\tilde{\mathbf{S}}$ and the enhanced strain field $\tilde{\mathbf{E}}$

$$\int_{\mathcal{B}_0} \tilde{\mathbf{S}} : \tilde{\mathbf{E}} \, \mathrm{d}V = 0 \tag{17}$$

the weak form of the boundary value problem is reduced to a two field problem

$$\int_{\mathcal{B}_0} \frac{\partial W_{0S}}{\partial \hat{\mathbf{E}}} : \delta \mathbf{E} \quad \mathrm{d}V - \int_{\mathcal{B}_0} \rho_0 \hat{\mathbf{b}} \cdot \delta \mathbf{u} \quad \mathrm{d}V - \int_{\partial \mathcal{B}_0} \hat{\mathbf{t}} \cdot \delta \mathbf{u} \quad \mathrm{d}A + \int_{\mathcal{B}_0} \frac{\partial W_{0S}}{\partial \hat{\mathbf{E}}} : \delta \tilde{\mathbf{E}} \quad \mathrm{d}V = 0.$$
(18)

The nonlinear eq. (18) is solved iteratively within the finite element method. For this purpose one needs the corresponding linearization, see also [6] and [16]

$$D[\delta\Pi] \cdot (\Delta \mathbf{u}, \Delta \tilde{\mathbf{E}}) = \int_{\mathcal{B}_0} (\delta \mathbf{E} : \frac{\partial^2 W_{0S}}{\partial \hat{\mathbf{E}}^2} : \Delta \mathbf{E} + \frac{\partial W_{0S}}{\partial \hat{\mathbf{E}}} : \Delta \delta \mathbf{E}) \quad \mathrm{d}V \\ + \int_{\mathcal{B}_0} \delta \mathbf{E} : \frac{\partial^2 W_{0S}}{\partial \hat{\mathbf{E}}^2} : \Delta \tilde{\mathbf{E}} \quad \mathrm{d}V + \int_{\mathcal{B}_0} \delta \tilde{\mathbf{E}} : \frac{\partial^2 W_{0S}}{\partial \hat{\mathbf{E}}^2} : \Delta \mathbf{E} \quad \mathrm{d}V \qquad (19) \\ + \int_{\mathcal{B}_0} \delta \tilde{\mathbf{E}} : \frac{\partial^2 W_{0S}}{\partial \hat{\mathbf{E}}^2} : \Delta \tilde{\mathbf{E}} \quad \mathrm{d}V \quad .$$

Here, the linearized virtual Green–Lagrange strain tensor reads

$$\Delta \delta \mathbf{E} = \frac{1}{2} (\Delta \mathbf{g}_i \cdot \delta \mathbf{g}_j + \delta \mathbf{g}_i \cdot \Delta \mathbf{g}_j) \mathbf{G}^i \otimes \mathbf{G}^j \quad .$$
 (20)

In the following the stresses obtained by partial derivatives from the strain energy function are denoted by $\mathbf{S} = \partial W_{0S} / \partial \hat{\mathbf{E}}$.

5 Finite element formulation

In the first part of this section we introduce the formulation of a standard displacement type element. Hence certain modifications are necessary to reduce the locking effects. To avoid shear locking the transverse shear strains are approximated using the interpolation functions of Bathe and Dvorkin [21]. Artificial thickness strains can be avoided using the interpolation functions of Betsch and Stein [2]. Furthermore the enhanced strains are specified with a different number of independent parameters.

5.1 Displacement type formulation

According to the isoparametric concept we use the standard tri–linear shape functions for an eight–node solid element to interpolate the geometry of the reference and the current configuration

$$\mathbf{X}^{h} = \sum_{I=1}^{nel} N_{I}(\xi^{1}, \xi^{2}, \xi^{3}) \mathbf{X}_{I} \qquad \mathbf{x}^{h} = \sum_{I=1}^{nel} N_{I}(\xi^{1}, \xi^{2}, \xi^{3}) \mathbf{x}_{I}, \qquad (21)$$

with nel = 8 and

$$N_I(\xi^1,\xi^2,\xi^3) = \frac{1}{8}(1+\xi_I^1\xi^1)(1+\xi_I^2\xi^2)(1+\xi_I^3\xi^3).$$
(22)

Here, the index h is used to denote the finite element approximation. The convective base vectors follows from eq. (1)

$$\mathbf{G}_{i}^{h} = \sum_{I=1}^{nel} N_{I,i} \mathbf{X}_{I} \qquad \mathbf{g}_{i}^{h} = \sum_{I=1}^{nel} N_{I,i} \mathbf{x}_{I}$$
(23)

and the approximation of the virtual strains is given by

$$\delta \mathbf{E}^{h} = \sum_{I=1}^{nel} \mathbf{B}_{I} \delta \mathbf{v}_{I} \qquad \mathbf{B}_{I} = [\mathbf{B}_{I}^{m}, \mathbf{B}_{I}^{s}]^{T} \quad .$$
(24)

Here, $\delta \mathbf{v}_I$ denotes the virtual nodal displacement vector where the components are given with respect to the fixed Cartesian basis system. The matrices \mathbf{B}_I^m and \mathbf{B}_I^s are specified below. The expression $\mathbf{S} : \Delta \delta \mathbf{E}^h$ leads to the geometrical matrix \mathbf{G}_{IJ} , where the linearized virtual strains $\Delta \delta \mathbf{E}$ are given in (20)

$$\mathbf{S} : \Delta \delta \mathbf{E}^{h} = \sum_{I=1}^{nel} \sum_{J=1}^{nel} \delta \mathbf{v}_{I}^{T} \mathbf{G}_{IJ} \Delta \mathbf{v}_{J} \quad \text{with} \quad \mathbf{G}_{IJ} = \text{diag} \left[\hat{S}_{IJ}, \, \hat{S}_{IJ}, \, \hat{S}_{IJ} \right] \quad . \tag{25}$$

The expression $\hat{S}_{IJ} = \hat{S}_{IJ}^m + \hat{S}_{IJ}^s$ results from two parts which are specified below.

5.2 Shear stiffness part

According to Fig. 3, four collocation points M = A, B, C, D with given coordinates ξ^i are defined.

Figure 3: Collocation points of the shear strain interpolation

At these points, the shear strains E_{13}^M , E_{23}^M of the Green–Lagrangean strain tensor are evaluated. To avoid shear locking, the transverse shear strains E_{13} and E_{23} are given using the interpolation functions introduced in [21]

$$\begin{bmatrix} 2E_{13}^h \\ 2E_{23}^h \end{bmatrix} = \begin{bmatrix} (1-\xi^2)E_{13}^B + (1+\xi^2)E_{13}^D \\ (1-\xi^1)E_{23}^A + (1+\xi^1)E_{23}^C \end{bmatrix} .$$
(26)

According to (26) the transverse shear strains are assumed to be constant in thickness direction within the considered element. Numerical tests showed that this approximation is sufficient for thin structures. The alternative with two planes and eight collocation points within the element leads not to significant differences. Hence the variation of the transverse shear strains can be expressed as

$$\begin{bmatrix} 2\delta E_{13}^h\\ 2\delta E_{23}^h \end{bmatrix} = \sum_{I=1}^{nel} \mathbf{B}_I^s \,\delta \mathbf{v}_I \tag{27}$$

with

$$\mathbf{B}_{I}^{s} = \frac{1}{2} \begin{bmatrix} (1-\xi^{2})(\mathbf{g}_{3}^{B^{T}}N_{I,1}^{B} + \mathbf{g}_{1}^{B^{T}}N_{I,3}^{B}) + (1+\xi^{2})(\mathbf{g}_{3}^{D^{T}}N_{I,1}^{D} + \mathbf{g}_{1}^{D^{T}}N_{I,3}^{D}) \\ (1-\xi^{1})(\mathbf{g}_{3}^{A^{T}}N_{I,2}^{A} + \mathbf{g}_{2}^{A^{T}}N_{I,3}^{A}) + (1+\xi^{1})(\mathbf{g}_{3}^{C^{T}}N_{I,2}^{C} + \mathbf{g}_{2}^{C^{T}}N_{I,3}^{C}) \end{bmatrix}$$
(28)

The shape function N_I^M and the current base vectors \mathbf{g}_i^M are obtained by exploitation of the corresponding equation at the collocation points M. The above defined quantity \hat{S}_{IJ}^s reads

$$\hat{S}_{IJ}^{s} = \frac{1}{2} [(1-\xi^{2})(N_{I,1}^{B} \ N_{J,3}^{B} + N_{I,3}^{B} \ N_{J,1}^{B}) + (1+\xi^{2})(N_{I,1}^{D} \ N_{J,3}^{D} + N_{I,3}^{D} \ N_{J,1}^{D})] \ S^{13} \\
+ \frac{1}{2} [(1-\xi^{1})(N_{I,2}^{A} \ N_{J,3}^{A} + N_{I,3}^{A} \ N_{J,2}^{A}) + (1+\xi^{1})(N_{I,2}^{C} \ N_{J,3}^{C} + N_{I,3}^{C} \ N_{J,2}^{C})] \ S^{23} .$$
(29)

5.3 Approximation of the thickness strains

For thin shell structures with bending dominated loading a locking effect due to artificial thickness strains has been observed by Ramm et al. in [23] when using a direct interpolation of the director vector. To overcome this locking effect an ANS-interpolation of the thickness strains E_{33} using bi-linear shape functions for four-node shell elements have been proposed by Betsch, Stein in [2] and by Bischoff, Ramm in [4]. Here, we adapt this procedure to the eight-node brick element. According to Fig. 4 four collocation points $L = A_1, A_2, A_3, A_4$ are defined in the reference surface with $\xi^3 = 0$.

The approximation of E_{33} reads

$$E_{33}^{h} = \sum_{L=1}^{4} \frac{1}{4} (1 + \xi_{L}^{1} \xi^{1}) (1 + \xi_{L}^{2} \xi^{2}) E_{33}^{L} \qquad L = A_{1}, A_{2}, A_{3}, A_{4}$$
(30)

where E_{33}^L denotes the thickness strains at the above defined points L. Thus with (30) it is assumed, that within the considered element E_{33} is constant in ξ^3 -direction. This assumption holds for thin structures.

The variation of the thickness strains and the membrane strains are obtained from

$$\begin{pmatrix} \delta E_{11}^h \\ \delta E_{22}^h \\ \delta E_{33}^h \\ 2\delta E_{12}^h \end{pmatrix} = \sum_{I=1}^{nel} \mathbf{B}_I^m \ \delta \mathbf{v}_I \tag{31}$$

with

$$\mathbf{B}_{I}^{m} = \begin{pmatrix} \mathbf{g}_{1}^{T} N_{I,1} \\ \mathbf{g}_{2}^{T} N_{I,2} \\ \sum_{L=1}^{4} \frac{1}{4} (1 + \xi_{L}^{1} \xi^{1}) (1 + \xi_{L}^{2} \xi^{2}) \ (\mathbf{g}_{3}^{L})^{T} N_{I,3}^{L} \\ \mathbf{g}_{2}^{T} N_{I,1} + \mathbf{g}_{1}^{T} N_{I,2} \end{pmatrix} .$$
(32)

Furthermore, the above defined quantity \hat{S}_{IJ}^m yields

$$\hat{S}_{IJ}^{m} = S^{11} N_{I,1} N_{I,1} + S^{22} N_{I,2} N_{I,2} + S^{33} \sum_{L=1}^{4} \frac{1}{4} (1 + \xi_{L}^{1} \xi^{1}) (1 + \xi_{L}^{2} \xi^{2}) N_{I,3}^{L} N_{J,3}^{L} + S^{12} (N_{I,1} N_{I,2} + N_{I,2} N_{I,1}).$$
(33)

5.4 EAS - Interpolation

The membrane behaviour of the 3D-shell element can be essentially improved applying the enhanced assumed strain method, [8]. Within a Lagrangean formulation we consider a strain field as is given in (13). The enhanced part is expressed with respect to different base vectors

$$\tilde{\mathbf{E}} = \tilde{E}_{ij} \,\mathbf{G}^i \otimes \mathbf{G}^j = \frac{\det \mathbf{J}_0}{\det \mathbf{J}} \,\tilde{E}_{ij}^0 \,\mathbf{G}_0^i \otimes \mathbf{G}_0^j \,. \tag{34}$$

The vectors \mathbf{G}_0^i and matrix \mathbf{J}_0 are evaluated at the element center. The matrix \mathbf{J} contains the base vectors \mathbf{G}_i as follows

$$\mathbf{J} = \begin{bmatrix} \mathbf{G}_1, \ \mathbf{G}_2, \ \mathbf{G}_3 \end{bmatrix},\tag{35}$$

whereas \mathbf{J}_0 is written in terms of the corresponding vectors \mathbf{G}_0^i . The tensor components \tilde{E}_{ij}^0 are arranged in a vector $\tilde{\mathbf{E}}^0$ according to (7). From (34) we obtain the matrix representation

$$\tilde{\mathbf{E}} = \frac{\det \mathbf{J}_0}{\det \mathbf{J}} \ \mathbf{T}_E^0 \ \tilde{\mathbf{E}}^0 , \qquad (36)$$

where the transformation matrix \mathbf{T}_{E}^{0} may be expressed with the coefficients $t_{ik}^{0} = \mathbf{G}^{i} \cdot \mathbf{G}_{0}^{k}$ similar to (9).

We assume interpolations, discontinuous over element boundaries, of the form

$$\tilde{\mathbf{E}}^0 = \mathbf{M}(\xi^1, \xi^2, \xi^3) \ \boldsymbol{\alpha}^e \,. \tag{37}$$

Here, α^e is a vector of independent parameters and **M** the below given interpolation matrix. With eq. (17) orthogonality of the independent stress field $\tilde{\mathbf{S}}$ and the assumed strain field $\tilde{\mathbf{E}}$ is assumed. The patch test requires the representation of a constant stress state within an element. Thus, considering (36) and (37)

$$\int_{\mathcal{B}_0} \frac{\det \mathbf{J}_0}{\det \mathbf{J}} \, (\tilde{\mathbf{S}}^c)^T (\mathbf{T}_E^0 \ \mathbf{M} \ \boldsymbol{\alpha}^e) \, \mathrm{d}V = 0 \tag{38}$$

must hold, where $\tilde{\mathbf{S}}^c$ denotes the vector of constant stresses. With $dV = \det \mathbf{J} d\xi^1 d\xi^2 d\xi^3$ the interpolation matrix \mathbf{M} must fulfill

$$\int_{\mathcal{B}_0} \mathbf{M}(\xi^1, \xi^2, \xi^3) \, \mathrm{d}\xi^1 \, \mathrm{d}\xi^2 \, \mathrm{d}\xi^3 = 0 \,.$$
(39)

Here, interpolations with 5, 8 and 11 parameters are chosen as follows

which fulfill the constraint (39). Introducing $\tilde{\mathbf{M}}$ as

$$\tilde{\mathbf{M}} = \frac{\det \mathbf{J}_0}{\det \mathbf{J}} \mathbf{T}_E^0 \mathbf{M}$$
(41)

the following stiffness matrices associated with the element nodes I, J

$$\mathbf{K}_{eIJ} = \int_{\mathcal{B}_{0}^{e}} (\mathbf{B}_{I}^{T} \mathbb{C} \mathbf{B}_{J} + \mathbf{G}_{IJ}) \, \mathrm{d}V$$

$$\mathbf{L}_{eI} = \int_{\mathcal{B}_{0}^{e}} \tilde{\mathbf{M}}^{T} \mathbb{C} \mathbf{B}_{I} \, \mathrm{d}V \qquad \mathbf{H}_{e} = \int_{\mathcal{B}_{0}^{e}} \tilde{\mathbf{M}}^{T} \mathbb{C} \tilde{\mathbf{M}} \, \mathrm{d}V \qquad (42)$$

and the following vectors are defined

$$\mathbf{f}_{eI}^{int} = \int_{\mathcal{B}_0^e} \mathbf{B}_I^T \mathbf{S} \, \mathrm{d}V \qquad \mathbf{f}_{eI}^{ext} = \int_{\mathcal{B}_0^e} N_I \,\rho_0 \, \hat{\mathbf{b}} \, \mathrm{d}V + \int_{\partial \mathcal{B}_0^e} N_I \hat{\mathbf{t}} \, \mathrm{d}A$$
$$\mathbf{h}_e = \int_{\mathcal{B}_0^e} \tilde{\mathbf{M}}^T \mathbf{S} \, \mathrm{d}V.$$
(43)

Here, **S** denotes the vector of stresses evaluated from the material law. Hence, the discretized linearized weak form yields with (19) and (20) the following system of equations on element level

$$\begin{bmatrix} \mathbf{K}_{e} & \mathbf{L}_{e}^{T} \\ \mathbf{L}_{e} & \mathbf{H}_{e} \end{bmatrix} \begin{bmatrix} \Delta \mathbf{v}^{e} \\ \Delta \boldsymbol{\alpha}^{e} \end{bmatrix} = \begin{bmatrix} \mathbf{f}_{e}^{ext} - \mathbf{f}_{e}^{int} \\ -\mathbf{h}_{e} \end{bmatrix}$$
(44)

Here, \mathbf{K}_e , \mathbf{L}_e , \mathbf{f}_e^{int} and \mathbf{f}_e^{ext} contains the submatrices \mathbf{K}_{eIJ} , \mathbf{L}_{eI} , \mathbf{f}_{eI}^{int} and \mathbf{f}_{eI}^{ext} according to the order of the nodes I and J. Furthermore $\Delta \mathbf{v}^e$ denotes the vector of the incremental element displacements. Since the enhanced strains are interpolated discontinuously across the element boundaries the parameters $\Delta \boldsymbol{\alpha}^e$ can be eliminated by static condensation on element level. This leads to the reduced problem $\mathbf{K}_{Te}\Delta \mathbf{v}_e = \mathbf{R}_e$ with the element matrices

$$\mathbf{K}_{Te} = \mathbf{K}_e - \mathbf{L}_e^T \mathbf{H}_e^{-1} \mathbf{L}_e \qquad \mathbf{R}_e = \mathbf{L}_e^T \mathbf{H}_e^{-1} \mathbf{h}_e + \mathbf{f}_e^{ext} - \mathbf{f}_e^{int}.$$
(45)

After assembly one obtains a pure displacement problem with the unknown nodal displacements.

6 Integration through the layers

The evaluation of the stiffness matrix and residual load vector is performed using a numerical Gauss integration. For the 8-node brick element with tri-linear shape functions two integration points are sufficient for each direction. This yields an exact integration of rectangular elements with a parallel epipetic geometry. Here, in total eight integration points are used for each layer of the laminate. According to (22) the element geometry is interpolated with tri-linear shape functions. After the first isoparametric map, see Fig. 5, we introduce a second isoparametric map for each layer.

Figure 5: First isoparametric map for the element geometry

The coordinates $\boldsymbol{\xi} = [\xi^1, \, \xi^2, \, \xi^3]^T$ are interpolated as follows

$$\boldsymbol{\xi} = \sum_{i=1}^{nlay} \bar{N}_i \quad \boldsymbol{\xi}_i \quad , \qquad \text{with} \qquad \bar{N}_i = \frac{1}{8} (1 + r^1 \ r_i^1) (1 + r^2 \ r_i^2) (1 + r^3 \ r_i^3) \tag{46}$$

where *i* represents the node number and $\boldsymbol{\xi}_i$ contains the coordinates of the considered layer. The coordinates $\mathbf{r} = [r^1, r^2, r^3]^T$, with $-1 \leq r^i \leq +1$ are defined with a second isoparametric space, see Fig. 6.

Figure 6: Second isoparametic map for the layer geometry

To evaluate the element matrices (42) and (43) we have to sum over all layers *nlay* and over all integration points *ngaus*. As example, the integration of the element stiffness matrix reads

$$\mathbf{K}_{e\,IJ} = \sum_{L=1}^{nlay} \sum_{igaus=1}^{ngaus} \left[\mathbf{B}_{I}^{T}(\boldsymbol{\xi}_{gp}^{L}) \, \mathbb{C}_{L} \, \mathbf{B}_{J}(\boldsymbol{\xi}_{gp}^{L}) + \mathbf{G}_{IJ}(\boldsymbol{\xi}_{gp}^{L}) \right] \, \det \mathbf{J}(\boldsymbol{\xi}_{gp}^{L}) \, \det \mathbf{J}^{L}(\mathbf{r}_{gp}^{L}) \, w_{gp}^{L} \, . \tag{47}$$

Here, **J** and \mathbf{J}^L denote the Jacobian matrix of the first and second map, respectively. Furthermore, w_{qp} are the weighting factors of the considered integration point.

7 Numerical examples

Subsequently we show the behaviour of the developed brick-type finite element within linear and nonlinear applications. In the following different element types based on the enhanced assumed strain method (EAS) and the assumed natural strain method (ANS) are distinguished. The used abbreviations to denote the element types are given in Table 1. As example, Q1A2E5 denotes a standard displacement element with additional assumed shear strain interpolation and five enhanced strain parameters. The Q1E30-element is formulated with respect to a global Cartesian basis system and possesses 30 enhanced parameters for all strain components, see [16]. The first two examples show the element behaviour with distorted meshes for plate and membrane problems. The last two examples are concerned with thin layered shells. The discretization is performed with only one element in thickness direction.

Q1	Displacement based element with tri–linear shape functions
A1	ANS with E_{33}
A2	ANS with E_{13} and E_{23}
A3	ANS with E_{13} , E_{23} and E_{33}
E5, E8, E11	EAS with $5,8$ or 11 parameters according to (40)
E30	EAS with all strain components, global Cartesian basis system

Table 1: Specification of abbreviations to denote the element types

7.1 Clamped plate with single load

With the following linear example the behaviour of the described elements is tested for a plate problem with distorted meshes. A clamped quadratic plate according to Fig. 7 is investigated. The length of the plate is L = 100 and the thickness is h = 1. The isotropic material behaviour is described with Young's modulus $E = 10^4$ and Poisson's ratio $\nu = 0.3$. The plate is loaded at the center by a single load F = 16.367. An analytical solution by Timoshenko and Woinowsky–Krieger [24] yields the center displacement as w = 1. Considering symmetry one quarter of the plate is dicretized with four elements and one element through the thickness. Here, we evaluate the center deflection with distorted meshes according to Fig. 7. The results for the element types are depicted in Fig. 8

Figure 7: Clamped plate with single load

for different values of the distortion parameter s. The diagram shows that the Q1E30– element yields good results for non-distorted meshes, whereas with increasing distortion parameter the element locks. This phenomenon has been also observed by Andelfinger [25] using plane stress shell elements. It shows, that the Q1E30–element can only be used for plate problems with regular meshes. The Q1A2–element with assumed strain interpolation for the transverse shear strains provides better results for distorted meshes. However it suffers from another locking effect which follows form the three–dimensional material law along with Poisson 's ratio not equal zero. This effect has been discussed for a shell formulation with an extensible director vector in [1]. In this example the elements Q1A2E5 and Q1A2E11 are locking free. Here, the essential improvement results from the enhanced thickness strains. For the present coarse mesh with four elements these results are obviously the best which can be achieved within the considered class of elements (eight– node brick elements or four–node shell elements with bilinear shape functions). The ANS interpolation of the thickness strains does not improve the behaviour of the element for this example.

Figure 8: Clamped plate with single load

7.2 Clamped beam subjected to couple forces

With this geometrical linear example the membrane behaviour of the different element types is investigated. Fig. 9 shows a clamped beam subjected to couple forces with F = 1000 at the free end. Two elements are used to discretize the cantilever. The discretization is chosen in such a way that the elements are subjected to inplane-bending. The material is isotropic with E = 1500 and $\nu = 0$.

Figure 9: Clamped beam subjected to couple forces

Fig. 10 shows the normalized displacement v versus the distortion parameter s. As can be seen the standard displacement element Q1 is obviously to stiff. Good results are obtained with the elements Q1E5 and Q1E8. The interpolation functions of the Q1E8– element contains a bilinear part $\xi^1\xi^2$ for the membrane strains E_{11} , E_{22} and E_{12} , see eq. (40). This is the reason for the slightly better results of the Q1E8–element in comparison with the element Q1E5 for highly distorted meshes. Here, the ANS modifications have no influence on the calculated displacements.

Figure 10: Clamped beam subjected to couple forces

7.3 Clamped cylindrical shell segment

A clamped layered cylindrical shell segment subjected to a single load $F = \lambda F_0$ with $F_0 = 200$ at the free end is considered next, see Fig. 11. The length of the cylinder is L = 304.8, the inner radius $R_i = 100.1$ and the thickness t = 3.0. A corresponding example with isotropic material was investigated by Stander, Matzenmiller and Ramm [26]. Here, we consider a composite material with $0^{\circ}/90^{\circ}/0^{\circ}$ and $90^{\circ}/0^{\circ}/90^{\circ}$ layer sequences, where zero degree and 90 degrees refer to the axial and the circumferential direction of the cylinder, respectively. The data for transversal isotropic material behaviour are given in Fig. 11. Here, the ratios ν_{12} , ν_{13} , ν_{23} and the moduli G_{12} , G_{13} , G_{23} are equal ν and G, respectively.

Figure 11: Clamped cylindrical shell segment

Considering symmetry conditions only half of the system is discretized. We use a finite element mesh with 16 elements in axis direction, 16 elements in circumferential direction and 1 element in thickness direction. The load deflection curves of the geometrical nonlinear problem are shown in Fig. 12 for the two different laminates. As the diagram shows, there is good agreement over the entire range of deformation between the results obtained with a four-node shell element [27] and the new brick-type shell element Q1A3E5.

Figure 12: Load deflection curve of the cylindrical shell segment

In the following we discuss the influence of the ANS and EAS interpolations. Fig. 13 shows load deflection curves for the layer sequence $90^{\circ}/0^{\circ}/90^{\circ}$ evaluated with different element types. According to Fig. 12 the curve obtained with the Q1A3E5–element can be seen as reference solution.

Figure 13: Load deflection curves for the layer sequence $90^{\circ}/0^{\circ}/90^{\circ}$

The Q1A1E5–element without ANS–interpolation for the transverse shear strains is to stiff even in the linear range. The Q1A3–element without enhanced strain formulation leads to a correct deflection v within the linear theory. However, with increasing load a locking effect can be observed when computing the nonlinear load deflection curve. A comparison of the results obtained with the Q1A2E5–element and the Q1A3E5–element shows that the ANS–interpolation for the thickness strains improves the behaviour of the element in the range of finite deformations. Furthermore the example shows that the EAS–method applied to the membrane strains and thickness strains is less important than the ANS– interpolation for the shear strains and thickness strains. However this holds only for this special structure.

Figure 14: Cylindrical shell segment with distorted mesh

Next we consider a distorted mesh according to Fig. 14. Here, we compare the Q1E30– element with 30 enhanced assumed strain parameters for all strain components (Cartesian formulation, see Ref.[16]) with the Q1A3E5–element. The associated load–deflection curves along with a reference solution evaluated with the Q1A3E5–element and a non– distorted mesh are depicted in Fig. 15. As can be seen the Q1A3E5–element is fairly insensitive with respect to mesh distortion whereas the Q1E30–element locks for distorted meshes. Finally, the deformed cylinder with a deflection v=160 of the loading point is shown in Fig. 16. It can be seen that finite deformations occur. Figure 15: Load deflection curves for the $90^{\circ}/0^{\circ}/90^{\circ}$ -laminate

Figure 16: Deformed cylindrical shell segment at v=160, layer sequence $90^{\circ}/0^{\circ}/90^{\circ}$

7.4 Buckling of a composite panel

Finally we investigate the buckling behaviour of a cylindrical carbon fiber reinforced composite panel. Associated experiments with different thickness parameters for the skin have been carried out within an ESA contract by the Institute of Structural Mechanics of the "Deutsche Zentrum für Luft- und Raumfahrt" (DLR) in Braunschweig, FRG, see also Ref. [28]. The panel consists of a cylinder segment with radius of the middle surface R = 340 mm and six stiffeners, which are glued at the inner side of the skin, see Fig. 17. The skin consists of 8 layers with a total thickness H = 1 mm.

Figure 17: Geometry and material data; rear view and top view of the panel

The cross-section of a stiffener is depicted in Fig. 18. The blade and the flange consist of 24 layers and 12 to 2 layers, respectively. The layup of the skin and stiffener is symmetric. The complicated layup of the stiffener flange is simplified within the finite element model according to Fig. 19. The layer sequence is given in Table 2, where zero degree refers to the axial direction of the cylinder segment. The layer thickness is h = 0.125 mm.

Figure 18: Cross–section of a stiffener

Figure 19: Simplified finite element mesh of a stiffener

	stiffener (blade)	stiffener (flange)	skin
layer sequence	$[(\pm 45^{\circ})_3; 0_6^{\circ}]_{sym}$	$[(\pm 45^{\circ})_3]$	$[90^{\circ}; \pm 45^{\circ}; 0^{\circ}]_{sym}$

Table 2: Layup of the panel in FE–discretization

The finite element discretization with Q1A3E5–elements is depicted in Fig. 20. The skin is discretized with one element through the thickness, 84 elements in length direction (80

elements for the inner range and 2 elements for the clamped range) and 66 elements in circumferential direction. The discretization of a stiffener can be seen in Fig. 19.

The boundary conditions are taken from the experiments and set as follows. The panel is clamped within a length of 90 mm at both ends of the skin, where deflection in axis direction is possible. At the lower edge the panel is supported in axis direction. Furthermore, the radial displacement at the straight edges is set to zero.

Figure 20: Load deflection curve of the panel

The panel is compressed at the top in axial direction where the axial displacement at the upper edge remains constant. The results of the numerical investigation based on a linear elastic model are depicted in Fig. 19. As can be seen, the load deflection behaviour obtained with the finite element model is practically linear. The load is increased up to a critical load $F = 123.0 \, kN$. The associated first and second eigenvectors are shown in Fig. 21. The plots show global buckling modes of the skin. One should note, that the eigenvalues lie very close together. Thus, a small variation of any geometrical parameter or of the finite element model may change the shape of the eigenvectors. This holds especially when imperfections are taken into account. The stability behaviour of the panel is fairly sensitive with respect to geometrical imperfections of the skin. However due to missing data, this is not considered within the present finite element discretization. The agreement between the numerical results and the experimental results especially the buckling load is good. The deviations in the upper range of the load deflection curve follow from inelastic effects which are not contained in the present finite element model. Finally Fig. 22 shows the panel in the experimental buckling state.

Figure 21: Radial component of the first and second eigenvector

Figure 22: Experimental buckling state of the carbon fiber reinforced panel

Acknowledgment: We thank Dr. R. Zimmermann of the DLR Braunschweig for the experimental results of the panel and the photography in Fig. 22.

8 Conclusion

In this paper a continuum based shell element is presented. The eight nodes are located at the surface of the element where each node possesses three displacement degrees of freedom. This allows to account for special boundary conditions like surface loads. Furthermore the displacement degrees of freedom are updated in a simple additive way within the Newton iteration process. The stresses are evaluated using a three–dimensional material law.

The numerical tests showed that the ANS–interpolations for the transverse shear strains and the thickness strains are essential for a locking–free element behaviour. Especially the assumed transverse shear interpolation avoids effectively shear locking even with distorted meshes. Here, it is essential that the element orientation is considered within the mesh generation. Due to the elimination process of the enhanced parameters the computing time increases with increasing number of internal parameters. Furthermore one can state that many enhanced parameters lead to a loss of robustness within the nonlinear solution processes especially for inelastic computations. The examples show that the versions with 8 or 11 enhanced strain parameters compared with 5 parameters yield only minor improvements of the element behaviour. Therefore the Q1A3E5–element is recommended for the nonlinear analysis of the considered thin–walled shell structures. The presented linear and nonlinear examples for layered composite shells show that the present brick–type element yields the same accurate results as a comparable four–node shell element. Here a discretization with only one element in thickness direction is used. However, applying a sufficient fine discretization through the thickness a correct description of the interlaminar shear and normal stresses is possible. This is e.g. necessary for a subsequent delamination analysis. A standard shell element which is formulated with respect to a reference surface does not offer this possibility.

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 $A = \begin{pmatrix} -1, & 0, & 0 \end{pmatrix}$ $B = \begin{pmatrix} 0, & -1, & 0 \end{pmatrix}$ $C = \begin{pmatrix} 1, & 0, & 0 \end{pmatrix}$ $D = \begin{pmatrix} 0, & 1, & 0 \end{pmatrix}$

Figure 3: Collocation points of the shear strain interpolation



- 0) 0)
 - 0)
 - 0)



Figure 5: First isoparametric map for the element geometry



Figure 6: Second isoparametic map for the layer geometry



Figure 7: Clamped plate with single load







Figure 10: Clamped beam subjected to couple forces





Figure 12: Load deflection curve of the cylindrical shell segment



Figure 13: Load deflection curves for the layer sequence $90^\circ/0^\circ/90^\circ$





Figure 15: Load deflection curves for the $90^{\circ}/0^{\circ}/90^{\circ}$ -laminate

















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